

UNCERTAINTY RELATIONS AS HILBERT SPACE GEOMETRY

Samuel L. Braunstein

*Department of Physics, Technion—Israel Institute of Technology,
32 000 Haifa, Israel*

Abstract

Precision measurements involve the accurate determination of parameters through repeated measurements of identically prepared experimental setups. For many parameters there is a “natural” choice for the quantum observable which is expected to give optimal information; and from this observable one can construct an Heisenberg uncertainty principle (HUP) bound on the precision attainable for the parameter. However, the classical statistics of multiple sampling directly gives

us tools to construct bounds for the precision available for the parameters of interest (even when no obvious natural quantum observable exists, such as for phase, or time); it is found that these direct bounds are more restrictive than those of the HUP. The implication is that the natural quantum observables typically do not encode the optimal information (even for observables such as position, and momentum); we show how this can be understood simply in terms of the Hilbert space geometry.

Another striking feature of these bounds to parameter uncertainty is that for a large enough number of repetitions of the measurements *all* quantum states are “minimum uncertainty” states — not just Gaussian wave-packets. Thus, these bounds tell us what precision is achievable as well as merely what is allowed.

1 Introduction

I want to start by pointing out that the very term “precision measurement” implicitly refers to an idealization of our use of quantum theory. It assumes that there are some c-number parameters (ordinary commuting numbers instead of operators) “out there” in the laboratory which have no, or at least negligible, quantum uncertainty for their values. For instance, in the case of the precision interferometric measurement of some path difference it is thus assumed that the end-mirror masses are so heavy that the initial uncertainty in their location and the quantum “diffusion” of their position throughout the entire measurement procedure are negligible. In this talk I will discuss the fundamental limits quantum theory places on our ability to determine c-number parameters through measurement. We might, for instance, be interested in the precision determination of fundamental constants such as the gravitational coupling G , the speed of light c , etc., or of laboratory parameters such as a time difference T , some path difference D , a phase difference Φ , etc. These latter parameters are just settings on some machine in our laboratory.

The type of machine we envision is one which has an inexhaustible supply of raw material which it uses to make a “train” of identical quantum states. These states will have encoded in them in some way the value of the parameter we seek. Thus, if our machine has some setting

which controls the value of a parameter X , then it will generate an inexhaustible supply of states $|\psi_X\rangle$; alternatively, if the setting is X' then it will generate a supply of new states $|\psi_{X'}\rangle$. Our job is to use some observable $\hat{\xi}$ to determine the parameter's value.

Let us consider a simple example where our machine sets some “location” parameter X which determines the location of the center of the peak of the wavefunctions generated:

$$\psi_X(x_i) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left[-\frac{(x_i - X)^2}{4\sigma^2}\right],$$

where i is a sequential label for replicas of our state generated by the laboratory machine at some fixed setting X . Because we are seeking the “location” of the state we might expect that the natural observable of position $\hat{\xi} = \hat{x}$ is optimal, so this is the observable we shall use. In seeking fundamental limits to measurement in quantum theory it is worth seeing what the Heisenberg uncertainty principle (HUP) says. For a single measurement we would simply take the value of position measured and use that as our estimate of the location parameter X of our state. In this way, the uncertainty for the parameter X would just be the uncertainty in our observable for this state, so

$$\Delta X = \Delta x \geq \frac{\hbar}{2\Delta p},$$

by the HUP, for a single measurement. Alternatively, for N measurements (each on a new replica of our state) we might take the average location we observe $\hat{x} = (\hat{x}_1 + \dots + \hat{x}_N)/N$ as an estimate for our parameter, which is an operator with conjugate momentum $\hat{p} = \hat{p}_1 + \dots + \hat{p}_N$ so the N -shot HUP will read

$$\Delta X = \Delta x \geq \frac{1}{\sqrt{N}} \frac{\hbar}{2\Delta p_1},$$

in terms of the uncertainty Δp_1 for a single state. This result shows a simple $1/\sqrt{N}$ improvement over the single-measurement HUP.

To summarize, we have learned that precision measurements deal with the language of c-number parameters — this being an idealization. We have learned that we must make some choice of observable to determine our parameter of interest, and where it exists we might choose the natural observable. Finally, we saw that the process of extraction of an estimate for our parameter from observation requires some form of data analysis — even if it be only the completely transparent procedure of using the value observed.

2 Bounds to data analysis

The previous section considered a crude multiple measurement uncertainty principle. It is surprising that much more powerful versions may be obtained from well known classical bounds to the efficiency of performing data analysis. The simplest of such bounds is called the Cramér-Rao lower bound [1], which we shall derive here. (Helstrom [2] was the first to place this bound in quantum language and recognize its significance; in this paper we broaden our physical understanding of this bound.)

Consider an arbitrary kind of data analysis based upon the N observations $x_1, \dots, x_N$ which yields an estimate X_{est} for the parameter X via a function

$$X_{\text{est}} = X_{\text{est}}(x_1, \dots, x_N)$$

which is independent of the value of the parameter sought, and for which each of the observations x_i is “drawn” independently from the probability distribution

$$p(x|X) = |\psi_X(x)|^2 .$$

To ensure that this data analysis has some connection with our parameter we impose the condition that on average it yields the correct value for X , i.e.,

$$X = \int X_{\text{est}}(x_1, \dots, x_N) p(x_1|X) \dots p(x_N|X) dx_1 \dots dx_N .$$

By taking the derivative of this with respect to X we obtain

$$1 = \int \Delta X_{\text{est}} \sum_{i=1}^N \left(\frac{\partial}{\partial X} \ln p(x_i|X) \right) p(x_1|X) \dots p(x_N|X) dx_1 \dots dx_N ,$$

where $\Delta X_{\text{est}} \equiv X_{\text{est}} - X$. Then applying the Schwartz inequality yields the Cramér-Rao bound

$$\Delta X_{\text{est}} \geq \frac{1}{\sqrt{NF}} ,$$

with F the Fisher information given by

$$F \equiv \int \left(\frac{\partial}{\partial X} p(x|X) \right)^2 p(x|X) dx .$$

Before we look at the consequences for this theorem from classical data analysis, we shall state one more theorem which applies to the specific kind of data analysis called maximum likelihood (ML) analysis. Fisher’s theorem [3] states that as $N \rightarrow \infty$ ML analysis is asymptotically efficient (since it asymptotes towards the Cramér-Rao bound), i.e.,

$$\Delta X_{\text{est}} \longrightarrow \frac{1}{\sqrt{NF}} ;$$

thus, at least one method of data analysis may reach the classical bound when a large enough sample of data is used. What is not given by this theorem is just how large a data sample is required to achieves this.

The simplest implication of the Cramér-Rao bound for a quantum system involves measurements of multiple copies of the pure state

$$\psi_X(x) = \psi(x - X) = r(x - X) \exp[i\phi(x - X)/\hbar] ,$$

where X is a so-called translation parameter. The result may be directly obtained from the uncertainty in the momentum for this state

$$(\Delta p)^2 = \frac{\hbar^2}{4} F + (\Delta \phi')^2 , \tag{1}$$

yielding from the Cramér-Rao bound

$$\Delta X_{\text{est}} \geq \frac{\hbar}{2\sqrt{N}\sqrt{(\Delta p)^2 - (\Delta \phi')^2}} \geq \frac{\hbar}{2\sqrt{N}\Delta p}.$$

This result is rather startling, since it shows that the classical bound is *always* more restrictive than the quantum bound; even for $N = 1$ when we would expect to have no qualms about the usual HUP. Further, it directly yields a parameter uncertainty principle for an arbitrary kind of data analysis, and hence is more general than the N -shot HUP we discussed in the previous section. To see why the classical bound can be more restrictive than the HUP we turn now to the metric on Hilbert space.

3 Quantum and classical metrics

Consider two ensembles $|\psi_X\rangle$ and $|\psi_{X'}\rangle$ which differ only in their values of the parameter X . For pure states there is a natural metric, measuring the distance in Hilbert space between a pair of states; since pure states are just rays of unit length, any suitable function of the angle between these rays will determine a Hilbert space metric. In particular, we shall choose the Study-Fubini metric [4]

$$d(|\psi_X\rangle, |\psi_{X'}\rangle) = 1 - |\langle\psi_X|\psi_{X'}\rangle|^2;$$

as the states overlap more they become less distinguishable which is reflected in a decrease in the metric.

For the observable $\hat{\xi}$ the two ensembles we are studying can be completely characterized by the probability distributions $p(\xi|X)$ and $p(\xi|X')$ (p and p' respectively). We can thus construct a classical metric to behave like the quantum one with respect to the distinguishability of states. Classical distinguishability, however, is limited by the Cramér-Rao bound, so if the nearby ensembles differ in

the parameter X by δX then they are permitted to be distinguishable so long as $NF(p)(\delta X)^2 \geq 1$. This gives a natural classical metric for nearby ensembles as

$$d(p, p') = \frac{1}{4}F(p)(\delta X)^2 + \dots,$$

where the scale factor is chosen for convenience. This result is equivalent to the metric introduced by Wootters [5] to lowest order.

To compare these two metrics, classical and quantum, we shall write the states in a discrete basis (labeled by j)

$$\begin{aligned}\psi_X\rangle &= \sum_j j \sqrt{p_j} e^{i\varphi_j} |j\rangle \\ I\psi_{X'}\rangle &= \sum_j j \sqrt{p_j + \delta p_j} e^{i(\varphi_j + \delta\varphi_j)} |j\rangle.\end{aligned}$$

Then to second order we may calculate the quantum metric to be

$$\begin{aligned}d(|\psi_X\rangle, |\psi_{X'}\rangle) &= (\delta X)^2 \left[\frac{1}{4} \sum_j j p_j \left(\frac{\partial}{\partial X} \ln p_j \right)^2 + \sum_j j p_j \left(\frac{\partial \varphi_j}{\partial X} \right)^2 - \left(\sum_j j p_j \frac{\partial \varphi_j}{\partial X} \right)^2 \right] + \dots \\ &= (\delta X)^2 \frac{(\Delta p_X)^2}{\hbar^2} + \dots\end{aligned}$$

where $\hat{p}_X$ is the “momentum” operator generating infinitesimal translations in the parameter X . The first term in square brackets is just the classical metric. The second and third terms correspond to the dispersion in the rate of change of the wavefunction’s phase; this is similar to Eq. (1), and reduces to it exactly when X is a translation parameter.

This relationship shows that classical distances between nearby ensembles are shorter than the corresponding quantum distances. Because the ensembles are harder to distinguish classically, the Cramér-Rao bound is more restrictive than the HUP. The classical distances are shorter than the quantum ones. We might ask whether the greater distances inherent in the quantum metric are accessible at all for the purposes of precision determination of parameters? We shall see that the answer is typically yes.

4 Seeing a quantum distance

The quantum metric

$$d(|\psi\rangle, |\psi'\rangle) = d(p, p') + (\Delta\varphi')^2(\delta X)^2 ,$$

exceeds the classical one only by the variance of the rate of change of the phase of the wavefunction. Thus, by a suitable choice of basis the phase can be made trivial so that the classical part of the metric makes up the entire contribution. This may seem a little strange since we typically will start with what we thought was the “natural” observable for a particular parameter. Yet since in general the phase will give some non-zero contribution this argument tells us that natural observables are rarely optimal.

We can see this surprising effect quite simply for a squeezed state. For the purposes of this example we will scale the units of position and momentum so that the ordinary HUP reads $\Delta x \Delta p \geq 1$. Suppose we wish to find the location parameter X for the squeezed state

$$|\psi_X\rangle = \hat{D}(X)\hat{R}(\theta)\hat{S}(r)|0\rangle .$$

On a phase-space diagram this corresponds to the squeezed vacuum (squeezed by e^{-r} in along the x -axis) rotated counter-clockwise by θ radians, and then displaced along the positive x -axis by the distance X . The natural observable for the position of the state would be $\hat{\xi} = \hat{x}$. So our mean signal would be

$$\mathcal{S} \equiv \langle \hat{x} \rangle = X ,$$

i.e., a measurement of $\hat{x}$ directly yields an estimate for the parameter X . Further, the uncertainty (or “noise”) in this value after a single measurement will be

$$\mathcal{N} \equiv \Delta x = \sqrt{e^{2r} \sin^2 \theta + e^{-2r} \cos^2 \theta} ,$$

which includes the reduced noise from the squeezed quadrature, plus an admixture (for $\theta \neq 0$) from the amplified quadrature.

By contrast, suppose we use the observable $\hat{\xi} \equiv \hat{x}' = \hat{R}(\theta)\hat{x}\hat{R}^\dagger(\theta)$. The new signal will be on average

$$\mathcal{S}' \equiv \langle \hat{x}' \rangle = X \cos \theta ,$$

which is reduced by a trigonometric factor, and the uncertainty in this value after a single measurement will be simply

$$\mathcal{N}' \equiv \Delta x' = e^{-r} .$$

A comparison shows that the signal-to-noise ratio has been improved by using a non-natural observable with

$$\frac{\mathcal{S}'}{\mathcal{N}'} > \frac{\mathcal{S}}{\mathcal{N}} ,$$

for $e^{2r} \sin(2\theta) > 1$ (an improvement for all $\theta \neq 0$ is possible using a more complicated rotation angle for the new observable).

In general then, so long as we can choose a basis where the phases are trivial we can “see” the full quantum distance for the new observable. In this ideal observable we will have a bound of

$$\Delta X_{\text{est, opt}} \geq \frac{\hbar}{2\sqrt{N}\Delta p_X} ,$$

which if X is a translation parameter will be just our original N -shot HUP. Now, however, Fisher’s theorem tell’s us that this lower bound can always be achieved so long as we take a large enough sample size (i.e., $N \rightarrow \infty$) and that we use the maximum likelihood method for our data analysis. This means that for the proper choice of observable (typically not the natural one) *all* V quantum states are minimum uncertainty states! This removal of so-called natural observables from their central place in quantum theory has practical implications for the precision measurement of time, phase, etc., where natural observables for these measurements are somewhat problematic.

5 Conclusion

Precision measurements naturally assume a language involving the existence of c-number parameters. The bounds to classical data analysis form a natural extension of the Heisenberg uncertainty principle from operator dispersions to parameter uncertainties; and generalize it to allow arbitrary data analysis on the results of multiple measurements of identically prepared systems. In general, it is found that the “natural” observables do not yield maximal information about the parameters sought; however, with a suitable observable and using maximum likelihood data analysis all quantum states (not just gaussian wavepackets) are minimum uncertainty states for a large enough number of measurements.

References

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